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Generalized Random Energy Model II

N. K. Jana[1] and B. V. Rao[2]

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The formulae for the free energy, when the driving distributions in Generalized Random Energy Model (GREM) are of the form $Ce^{-|x|^\gamma}$ for $\gamma \geq 1$ are derived. The large deviation technique allows the use of different distributions at different levels of the GREM. As an illustration we consider, in detail, a two level GREM with exponential and Gaussian distributions. This simple case itself leads to interesting phenomena.

KEY WORDS: spin glasses, large deviation principle, free energy

1. INTRODUCTION

This note is essentially a continuation of Ref. 11. To quickly recall the set up, we have an integer $n \geq 1$ fixed once and for all. For $N(\geq n)$ particle system the configuration space is $\Sigma_N = \{+1, -1\}^N$, also denoted by 2^N. For each N, we have a partition $N = k_{1N} + \cdots + k_{nN}$, where each $k_{iN} \geq 1$ is an integer. Using the obvious correspondence $2^N = \prod_i 2^{k_{iN}}$, we write $\sigma \in 2^N$ as $\sigma_1\sigma_2 \cdots \sigma_n$ where $\sigma_i \in 2^{k_{iN}}$. We think of this Σ_N as an n level tree where the first level nodes are indexed by $\sigma_1 \in 2^{k_{1N}}$ and the second level nodes below σ_1 being denoted by $\sigma_1\sigma_2$ with $\sigma_2 \in 2^{k_{2N}}$ etc. For each level i, we fix a number $a_i > 0$ as weight for that level. For each $i \geq 1$ and each node $\sigma_1 \cdots \sigma_i$ of the i-th level we fix a symmetric random variable $\xi_{\sigma_1 \cdots \sigma_i}$. These random variables are independent. For a configuration $\sigma \in \Sigma_N$ the Hamiltonian is given by

$$H_N(\sigma) = \sum_{1 \leq i \leq n} a_i \xi_{\sigma_1 \cdots \sigma_i}.$$

[1] Department of Mathematics, Bijoy Krishna Girls' College, Howrah and Stat-Math Unit, Indian Statistical Institute, Kolkata, India; e-mail: nabin_r@isical.ac.in
[2] Stat-Math Unit, Indian Statistical Institute, Kolkata, India; e-mail: bvrao@isical.ac.in

For $\beta > 0$, the partition function is

$$Z_N(\beta) = 2^N \mathbf{E}_\sigma e^{\beta H_N(\sigma)}.$$

where $\mathbf{E}_\sigma$ is expectation with respect to σ when 2^N has uniform distribution. In other words, Z_N is simply the sum of all the Gibbs factors $e^{\beta H_N(\sigma)}$. Since our random variables are symmetric, we omitted the minus sign in the exponent. The object of interest is the free energy $\mathcal{E}(\beta) = \lim_N \frac{1}{N} \log Z_N(\beta)$.

In Ref. 11, we considered the model when all the random variables ξ_σ are identically distributed and are either two sided exponential or Gaussian. Explicit formulae for $\mathcal{E}(\beta)$ were given when for each i, $\frac{k_{iN}}{N} \to p_i$ as $N \to \infty$. This involves as usual, two steps. First is evaluating the rate function for an appropriate sequence of probabilities and then using Varadhan's lemma[4,11] to arrive at a variational formula. (Here we wish to point out that in Proposition 1 at the end of Sec. 2 in Ref. 11, one has to assume that the sequence $\{\mu_N\}$ is eventually supported on a compact set instead of assuming compact support for I.) Second is solving the variational problem to get an explicit formula. Though the first step could be done for other distributions, we could not solve the variational problem in Ref. 11. In Sec. 2, we fill this gap. It is worth noting that non-Gaussian distributions with exponentially decaying tails were considered in Ref. 9 for REM. A. Bovier and a referee have drawn our attention to Refs. 1, 2, where a deep and detailed study of the partition function was carried out—the Gaussian setup in Ref. 2 and more general Weibull setup in Ref. 1—again in case of REM. For the GREM, we do not have similar results.

But it is interesting to note that the large deviation technique allows one to use different driving distribution at different levels. The variational formula for the free energy still holds good. Explicit formulae, of course, depend on the choice of distributions. Though we do not have closed form expressions, in general, we consider two cases as illustration. We discuss two level GREM with exponential and Gaussian distributions. The resulting formulae exhibit some peculiarities. Perhaps these should be looked into more seriously. These are all considered in Sec. 3. Though mathematically it is quite alright to do this, the physics eludes us. Does this mean we are considering spin glasses where materials of diverse magnetic susceptibilities are present? We are not sure.

2. I.I.D. CASE

We fix a number $\gamma \geq 1$. In this section we consider an n level GREM where for the N particle system the random variables $\xi_{\sigma_1 \cdots \sigma_i}$ are i.i.d. having probability density

$$\phi_{N,\gamma}(x) = \text{Const. } e^{-\frac{|x|^\gamma}{\gamma N^{\gamma-1}}} \qquad -\infty < x < \infty,$$

More precisely,

$$\phi_{N,\gamma}(x) = \frac{1}{2\Gamma\left(\frac{1}{\gamma}\right)}\left(\frac{\gamma}{N}\right)^{\frac{\gamma-1}{\gamma}} e^{-\frac{|x|^\gamma}{\gamma N^{\gamma-1}}} \qquad -\infty < x < \infty.$$

Note that $\phi_{N,1}$ is independent of N and is two sided exponential density with parameter 1. On the other hand, $\phi_{N,2}$ is Gaussian density with mean 0 and variance N. Of course, γ can be larger than 2 as well.

If we define the map $\Sigma_N = \prod_i 2^{k_{iN}} \to \mathbb{R}^n$ by

$$\sigma \mapsto \left(\frac{\xi_{\sigma_1}(\omega)}{N}, \frac{\xi_{\sigma_1\sigma_2}(\omega)}{N}, \ldots, \frac{\xi_{\sigma_1\cdots\sigma_n}(\omega)}{N}\right)$$

and transport the uniform probability of Σ_N to $\mathbb{R}^n$, we get a probability $\mu_N(\omega)$ on $\mathbb{R}^n$. Proceeding as in Ref. 11, the sequence $\{\mu_N(\omega) : N \geq n\}$ satisfies, for a.e. ω, LDP with rate function I given by

$$I(\widetilde{x}) = \begin{cases} \dfrac{1}{\gamma}\displaystyle\sum_{i=1}^{n} |x_i|^\gamma & \text{if } \widetilde{x} \in \Psi \\ \infty & \text{otherwise} \end{cases}$$

where

$$\Psi = \left\{\widetilde{x} \in \mathbb{R}^n : \sum_{i=1}^{k} \frac{|x_i|^\gamma}{\gamma} \leq \sum_{i=1}^{k} p_i \log 2,\ \ 1 \leq k \leq n\right\},$$

and $p_i = \lim_{N\to\infty} \frac{k_{iN}}{N}$, which is assumed to exist. Again as in Ref. 11, by Varadhan's lemma,

$$\lim_N \frac{1}{N} \log Z_N(\beta) = \log 2 - \inf_{\widetilde{x}\in\Psi^+} \sum_{i=1}^{n}\left(\frac{x_i^\gamma}{\gamma} - \beta a_i x_i\right),$$

where Ψ^+ is Ψ intersected with the positive orthant of $\mathbb{R}^n$. The case $\gamma = 1$ and $\gamma = 2$ are precisely the exponential and Gaussian cases considered in Ref. 11. Henceforth we assume $\gamma > 1$. To evaluate the infimum let us put, for $1 \leq j \leq k \leq n$,

$$B(j,k) = \left(\frac{(p_j + \cdots + p_k)\gamma \log 2}{a_j^{\frac{\gamma}{\gamma-1}} + \cdots + a_k^{\frac{\gamma}{\gamma-1}}}\right)^{\frac{\gamma-1}{\gamma}}.$$

Set $r_0 = 0$ and for $l \geq 0$ (integer),

$$\beta_{l+1} = \min_{k>r_l} B(r_l+1,k) \quad r_{l+1} = \max\{i > r_l : B(r_l+1,i) = \beta_{l+1}\}.$$

Clearly, for some K with $1 \leq K \leq n$, we have $r_K = n$. Put $\beta_0 = 0$ and $\beta_{K+1} = \infty$. Note that $0 = \beta_0 < \beta_1 < \beta_2 \cdots < \beta_K < \beta_{K+1} = \infty$.

Fix $j \le K$ and let $\beta \in (\beta_j, \beta_{j+1}]$. Define $\tilde{\overline{x}} \in \Psi^+$ as follows:

$$\overline{x}_i = \begin{cases} (\beta_l a_i)^{\frac{1}{\gamma-1}} & \text{if } i \in \{r_{l-1}+1, \ldots, r_l\} \text{ for some } l,\ 1 \le l \le j \\ (\beta a_i)^{\frac{1}{\gamma-1}} & \text{if } i \ge r_j + 1 \end{cases}.$$

Claim: $\inf_{\tilde{x} \in \Psi^+} \sum_{i=1}^n \left(\frac{x_i^\gamma}{\gamma} - \beta a_i x_i \right)$ occurs at $\tilde{\overline{x}}$.

In order to prove the claim, fix any $\tilde{x} \in \Psi^+$. For $k \le j$, first note that, by Holder's inequality,

$$\sum_{i=1}^{r_k} x_i \overline{x}_i^{\gamma-1} \le \left(\sum_{i=1}^{r_k} x_i^\gamma \right)^{\frac{1}{\gamma}} \left(\sum_{i=1}^{r_k} \overline{x}_i^\gamma \right)^{\frac{\gamma-1}{\gamma}} \le \sum_{i=1}^{r_k} \overline{x}_i^\gamma.$$

where the last inequality follows by observing that $\tilde{x} \in \Psi^+$ so that $\sum_{i=1}^{r_k} x_i^\gamma \le \sum_{i=1}^{r_k} \gamma p_i \log 2 = \sum_{i=1}^{r_k} \overline{x}_i^\gamma$. Hence, $\sum_{i=1}^{r_k} \overline{x}_i^{\gamma-1}(\overline{x}_i - x_i) \ge 0$.

Since $\beta > \beta_j$, we have $(\frac{\beta}{\beta_l} - 1) > 0$ for $1 \le l \le j$. Moreover since β_l are increasing with l, these numbers $(\frac{\beta}{\beta_l} - 1)$ are decreasing. It follows that,

$$\sum_{l=1}^{j} \left(\frac{\beta}{\beta_l} - 1 \right) \sum_{i=r_{l-1}+1}^{r_l} \overline{x}_i^{\gamma-1}(\overline{x}_i - x_i) \ge 0.$$

In other words, using the definition of $\overline{x}_i$,

$$\sum_{i=1}^{r_j} \beta a_i (\overline{x}_i - x_i) \ge \sum_{i=1}^{r_j} \overline{x}_i^{\gamma-1}(\overline{x}_i - x_i). \tag{1}$$

Now,

$$\begin{aligned} &\sum_{i=1}^{r_j} \left(\frac{x_i^\gamma}{\gamma} - \beta a_i x_i \right) - \sum_{i=1}^{r_j} \left(\frac{\overline{x}_i^\gamma}{\gamma} - \beta a_i \overline{x}_i \right) \\ &= \sum_{i=1}^{r_j} \left(\frac{x_i^\gamma}{\gamma} + \beta a_i(\overline{x}_i - x_i) - \frac{\overline{x}_i^\gamma}{\gamma} \right) \\ &\ge \sum_{i=1}^{r_j} \left(\frac{x_i^\gamma}{\gamma} + \overline{x}_i^{\gamma-1}(\overline{x}_i - x_i) - \frac{\overline{x}_i^\gamma}{\gamma} \right) \qquad \text{by (1)} \\ &= \sum_{i=1}^{r_j} \left(\frac{x_i^\gamma}{\gamma} + \frac{\gamma-1}{\gamma} \overline{x}_i^\gamma - x_i \overline{x}_i^{\gamma-1} \right) \\ &\ge 0, \end{aligned} \tag{2}$$

where in the last inequality we used $x_i \overline{x}_i^{\gamma-1} \le \frac{1}{\gamma} x_i^\gamma + \frac{\gamma-1}{\gamma} \overline{x}_i^\gamma$.

On the other hand, utilizing the definition of $\overline{x}_i$ and the inequality $\beta a_i x_i \leq \frac{x_i^\gamma}{\gamma} + \frac{\gamma-1}{\gamma}(\beta a_i)^{\frac{\gamma}{\gamma-1}}$ we have,

$$\begin{aligned}
&\sum_{i=r_j+1}^{n}\left(\frac{x_i^\gamma}{\gamma} - \beta a_i x_i\right) - \sum_{i=r_j+1}^{n}\left(\frac{\overline{x}_i^\gamma}{\gamma} - \beta a_i \overline{x}_i\right)\\
&\quad = \sum_{i=r_j+1}^{n}\left(\frac{x_i^\gamma}{\gamma} + \frac{\gamma-1}{\gamma}(\beta a_i)^{\frac{\gamma}{\gamma-1}} - \beta a_i x_i\right)\\
&\quad \geq 0. \qquad (3)
\end{aligned}$$

Clearly, (2) and (3) complete proof of the claim. This argument is in fact a generalization of Dorlas and Dukes,[8] Capocaccia *et al.*[3]

All this leads to the following explicit formula for the free energy.

Theorem 1. *Almost surely,*

$$\lim_N \frac{1}{N}\log Z_N(\beta)$$
$$= \begin{cases} \displaystyle\sum_{i=r_j+1}^{n} p_i \log 2 + \frac{\gamma-1}{\gamma}\sum_{i=r_j+1}^{n}(\beta a_i)^{\frac{\gamma}{\gamma-1}} + \beta\sum_{l=1}^{j}\beta_l^{\frac{1}{\gamma-1}}\sum_{i=r_{l-1}+1}^{r_l} a_i^{\frac{\gamma}{\gamma-1}} & \text{if } \beta_j < \beta \leq \beta_{j+1},\ 0 \leq j \leq K-1\\ \displaystyle\beta\sum_{l=1}^{K}\beta_l^{\frac{1}{\gamma-1}}\sum_{i=r_{l-1}+1}^{r_l} a_i^{\frac{\gamma}{\gamma-1}} & \text{if } \beta > \beta_K \end{cases}$$

Observe that for $\gamma = 2$ this coincides with the well known formula, originally appearing as formula (23) in Ref. 7.

3. NON IDENTICALLY DISTRIBUTED CASE

It is worth noting that the LDP holds good even when the driving distributions at various levels are different. That is, fix numbers $\gamma_1, \ldots, \gamma_n$; each at least one and consider an n level GREM where the driving distribution at the i-th level is ϕ_{N,γ_i}. More precisely, for any node $\sigma_1 \cdots \sigma_i$ at the i-th level $\xi_{\sigma_1\cdots\sigma_i}$ has density ϕ_{N,γ_i}. Of course, all the random variables are independent. Define as earlier, the map $\Sigma_N \to \mathbb{R}^n$ by

$$\sigma \mapsto \left(\frac{\xi_{\sigma_1}(\omega)}{N}, \frac{\xi_{\sigma_1\sigma_2}(\omega)}{N}, \ldots, \frac{\xi_{\sigma_1\cdots\sigma_n}(\omega)}{N}\right).$$

Let $\mu_N(\omega)$ be the induced probability on $\mathbb{R}^n$ when Σ_N is equipped with uniform probability. The arguments in Ref. 11, with appropriate change, show that for a.e.

ω, the sequence of probabilities $\{\mu_N(\omega), N \geq n\}$ on $\mathbb{R}^n$ satisfies LDP with rate function I given by

$$I(\widetilde{x}) = \begin{cases} \sum_{i=1}^{n} \frac{|x_i|^{\gamma_i}}{\gamma_i} & \text{if } \widetilde{x} \in \Psi \\ \infty & \text{otherwise} \end{cases}$$

where

$$\Psi = \left\{ \widetilde{x} \in \mathbb{R}^n : \sum_{i=1}^{k} \frac{|x_i|^{\gamma_i}}{\gamma_i} \leq \sum_{i=1}^{k} p_i \log 2, \quad 1 \leq k \leq n \right\},$$

and, of course $p_i = \lim_{N\to\infty} \frac{k_{iN}}{N}$. Let, as earlier, Ψ^+ be the part of Ψ in the positive orthant of $\mathbb{R}^n$. Now we have the following:

Theorem 2. *If the driving distribution has density ϕ_{N,γ_i} at the i-th level, we have almost surely,*

$$\lim_N \frac{1}{N} \log Z_N(\beta) = \log 2 - \inf_{\widetilde{x} \in \Psi^+} \left\{ \sum_{i=1}^{n} \left(\frac{x_i^{\gamma_i}}{\gamma_i} - \beta a_i x_i \right) \right\}.$$

To get a better understanding of this expression, we now specialize to the case $n = 2$. The limiting frequencies $\lim_N \frac{k_{iN}}{N}$ are p_i for $i = 1, 2$. The weights for the two levels are a_1 and a_2 respectively. We assume p_1, p_2, a_1, a_2 are strictly positive.

(A) Exponential-Gaussian GREM:

In this case we consider the distributions at the first level to be $\phi_{N,1}$ and at the second level to be $\phi_{N,2}$—that is, exponential and Gaussian respectively. Then the formula above reads as follows:

$$\mathcal{E}(\beta) = \log 2 - \inf \left\{ f(x, y) : x, y \geq 0;\ x \leq p_1 \log 2;\ x + \frac{1}{2} y^2 \leq \log 2 \right\}$$

where

$$f(x, y) = x(1 - \beta a_1) + \frac{1}{2} y^2 - \beta a_2 y.$$

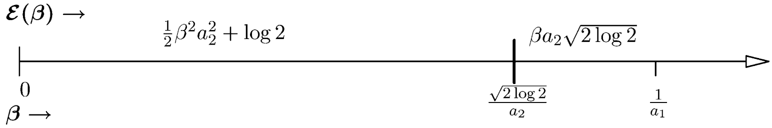
If $\beta \leq \frac{1}{a_1}$, then clearly this function attains its minimum at the point, $(0, \beta a_2 \wedge \sqrt{2 \log 2})$.

If $\beta > \frac{1}{a_1}$, here is how to calculate the infimum. The function $g(y) = \inf_x f(x, y)$ is given by

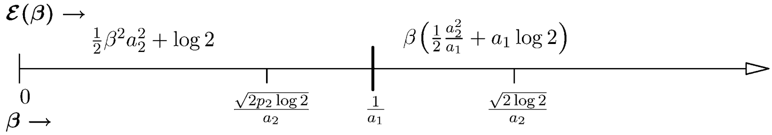
$$g(y) = \begin{cases} p_1(1 - \beta a_1)\log 2 + \frac{1}{2}y^2 - \beta a_2 y & \text{for } 0 \le y \le \sqrt{2p_2 \log 2} \\ (1 - \beta a_1)\log 2 + \frac{1}{2}\beta a_1 y^2 - \beta a_2 y & \text{for } \sqrt{2p_2 \log 2} \le y \le \sqrt{2 \log 2} \end{cases}$$

Since the required infimum of f is just the infimum of $g(y)$, one can calculate it, after some work, by analyzing g in the two intervals separately. This leads to the following three scenarios. We express $\mathcal{E}(\beta)$ against β as follows. The values below the line are values of β, where as above the line are of $\mathcal{E}(\beta)$. A phase transition occurs at the dark lines.

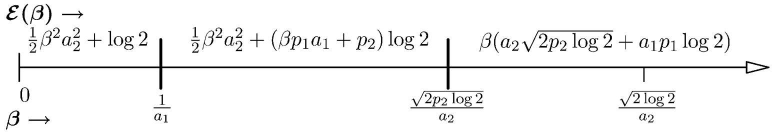
Subcase 1A



Subcase 2A



Subcase 3A



Thus in subcase 1A, the system behaves like a Random Energy Model (REM) with Gaussian distributions[5] having weight a_2, that is, as if $H_N(\sigma)$ are i.i.d centered Gaussian with variance $a_2^2 N$. For example, when $a_1 = a_2$ then this is just the standard Gaussian REM. It does not depend on the quantities p_1 and p_2. Even when $p_2 = 0.0001$ (very small) the first level exponentials do not show up in the limit. Further the GREM reduces to a REM. Of course, this is so as long as $\sqrt{2 \log 2} < \frac{a_2}{a_1}$.

Subcase 3A seems rather peculiar. This is indeed a regular GREM. Imagine placing exponential random variables ξ_{σ_1} at the first level and one i.i.d bunch $\{\xi_{\sigma_1 \sigma_2}\}$ is placed below each first level node. In other words, consider $\{\eta_{\sigma_2} : \sigma_2 \in 2^{k_{2N}}\}$ i.i.d $\mathcal{N}(0, N)$ and set $\xi_{\sigma_1 \sigma_2} = \eta_{\sigma_2}$ for all σ_1, σ_2. Consider the corresponding Hamiltonian $H_N(\sigma) = a_1 \xi_{\sigma_1} + a_2 \xi_{\sigma_1 \sigma_2} = a_1 \xi_{\sigma_1} + a_2 \eta_{\sigma_2}$. Let us set $Z_N^1 = \sum_{\sigma_1} e^{\beta a_1 \xi_{\sigma_1}}$, the partition function for the k_{1N}-particles system consisting of exponential Hamiltonians with weight a_1. Let $Z_N^2 = \sum_{\sigma_2} e^{a_2 \eta_{\sigma_2}}$, the partition function for k_{2N} particle system consisting of Gaussian, $\mathcal{N}(0, N)$ Hamiltonians with weight a_2. Clearly, $Z_N = Z_N^1 \cdot Z_N^2$. If, for $i = 1, 2$; $\mathcal{E}_i = \lim_N \frac{1}{N} \log Z_N^i$ then the exponential REM formula[10,11] yields,

$$\mathcal{E}_1(\beta) = \begin{cases} p_1 \log 2 & \text{if } \beta \leq \dfrac{1}{a_1} \\ \beta p_1 a_1 \log 2 & \text{if } \beta > \dfrac{1}{a_1} \end{cases} \tag{4}$$

The Gaussian REM formula (keeping in mind that for N fixed, the k_{2N} particle system has $\mathcal{N}(0, N)$ Hamiltonians as opposed to $\mathcal{N}(0, k_{2N})$) yields,

$$\mathcal{E}_2(\beta) = \begin{cases} p_2 \log 2 + \dfrac{1}{2} a_2^2 \beta^2 & \text{if } \beta \leq \dfrac{\sqrt{2 p_2 \log 2}}{a_2} \\ \beta a_2 \sqrt{2 p_2 \log 2} & \text{if } \beta > \dfrac{\sqrt{2 p_2 \log 2}}{a_2} \end{cases} \tag{5}$$

One can now verify that,

$$\mathcal{E}(\beta) = \mathcal{E}_1(\beta) + \mathcal{E}_2(\beta).$$

In other words the GREM behaves like sum of two independent REMs, one exponential and other Gaussian. The word independent is used here in the sense that there is no interaction between these two REMs—that is, there is no interaction between the k_{1N} particles and the k_{2N} particles, as if there is a barrier between these two sets of particles. Of course, this is so as long as $\frac{a_2}{a_1} < \sqrt{2 p_2 \log 2}$. This should be contrasted with subcase 1A where the entire system behaves like an N particle Gaussian REM.

Finally, coming to subcase 2A, we observe that the free energy up to $\frac{1}{a_1}$ is given by $\log 2 + \frac{1}{2} \beta^2 a_2^2$. This can be thought of as the Gaussian REM energy but not going all the way up to $\beta \leq \frac{\sqrt{2 \log 2}}{a_2}$ but cut short at $\frac{1}{a_1}$. This can also be thought of as the sum of the two energies $\mathcal{E}_1$ and $\mathcal{E}_2$ as in (4) and (5), but then the Gaussian effect is prolonged up to $\beta \leq \frac{1}{a_1}$ instead of stopping at $\frac{\sqrt{2 p_2 \log 2}}{a_2}$. We do not know which is the correct interpretation. For $\beta > \frac{1}{a_1}$, the system exhibits a new phenomenon which we are unable to explain. The term $\beta a_1 \log 2$ is reminiscent of the exponential REM energy for the N particle system. The other term $\frac{1}{2} \beta \frac{a_2^2}{a_1}$ is not reminiscent of anything we know.

(B) Gaussian-Exponential GREM: In this case, we consider the distributions at the first level to be Gaussian, $\phi_{N,2}$ where as at the second level they are exponential, $\phi_{N,1}$. As earlier p_i and a_i correspond to the i-th level. Now, the general formula of Theorem 2 reduces to the following:

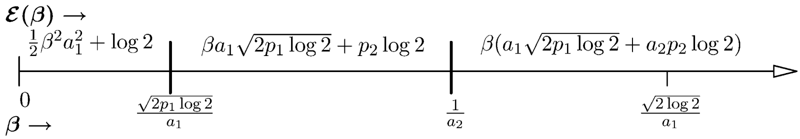
$$\widetilde{\mathcal{E}}(\beta) = \log 2 - \inf\left\{\widetilde{f}(x,y) : x, y \geq 0;\ x \leq \sqrt{2p_1 \log 2};\ \frac{1}{2}x^2 + y \leq \log 2\right\},$$

where

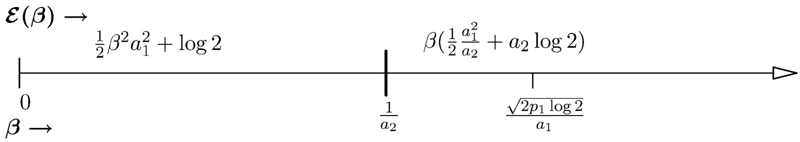
$$\widetilde{f}(x,y) = \frac{1}{2}x^2 - \beta a_1 x + y(1 - \beta a_2).$$

Analysis similar to (A), yields the following explicit formula for the free energy as a function of β. We follow the same convention as mentioned earlier.

Subcase 1B



Subcase 2B



Remarks similar to Case (A) apply here as well. The reader should note that to compare this case with case (A), one should interchange a_2 with a_1 and p_2 with p_1 (to maintain the same weights and proportions for the exponential and Gaussian levels). Subcase 1B is similar to subcase 2A, where as subcase 2B is similar to that of subcase 3A. In (B), the system never reduces completely to a Gaussian REM as happened in subcase 1A.

4. CONCLUSIONS

It is observed that the free energy exists in Generalized Random Energy Models even when the driving distributions are non-Gaussian. The large deviation technique allows the use of different distributions at different levels. As an illustration, we considered a 2 level GREM. The conclusions of Exponential-Gaussian GREM differ from those of Gaussian-Exponential. The system may reduce to a Gaussian REM or to two independent REMs.

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